Technical Comments

Comments on "Displacement Thickness for Boundary Layers with Surface Mass Transfer"

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FANNELOP[1,2] obtained expressions for the inviscid-viscous matching conditions for two-dimensional bodies and axisymmetric bodies with transverse curvature and for both bodies with surface mass transfer. The expressions he obtained did much to remove some ambiguities regarding the correct matching conditions. Unfortunately, in the case of axisymmetric bodies with transverse curvature, the "edge" of the boundary layer was used by Fannelop[1] in his definition of a mass-flow defect thickness δ_a^* given by Eq. (16) of Ref. 1. The arbitrarily defined edge of the boundary layer at $y = \delta$ can be removed easily if one introduces a new parameter y_∞, which is the physical distance normal to the surface where the outer boundary conditions are imposed on the boundary-layer equations. The conditions at this point outside the boundary layer are unambiguously defined to be the inviscid outer flow conditions when gradients in the outer flow normal to the surface are negligible. Fannelop's axisymmetric mass-flow defect thickness in terms of y_∞ would read

$$\delta_a^* = \int_0^{y_\infty} \left[1 - \frac{(r_w + y)\rho u}{(r_w + y_\infty)\rho_e u_e}\right] dy \tag{1}$$

which is not the axisymmetric displacement thickness with transverse curvature which Fannelop correctly states as

$$\int_0^{\delta^*} \rho_e u_e r \, dy = \int_0^{y_\infty} (\rho_e u_e - \rho u) r \, dy \tag{2}$$

where the upper limit δ has been replaced by y_∞.

The expression obtained by Fannelop for two-dimensional flow can be extended easily to include axisymmetric flow without transverse curvature. His Eq. (5a) would read

$$\Delta^* = \delta^* + \frac{1}{\rho_e u_e r_w{}^j} \int_0^x \rho_w u_w r_w{}^j \, dx \tag{3}$$

where

$$\delta^* = \int_0^{y_\infty} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \tag{4}$$

with $j = 0$ for two-dimensional flow and $j = 1$ for axisymmetric flow (see Ref. 1 for the definition of remaining quantities).

Numerical Examples

Lewis, Marchand, and Little[3] recently treated the effects of inviscid-viscous matching conditions for sharp cones at $M_\infty = 3.93$, 5.64, and 10. They considered the complete expression for v_e/u_e obtained by Li and Gross,[4] and they found that the terms in the expression for v_e/u_e neglected by Li and Gross had a significant effect on the matching condition.

Two of the conditions previously treated in Ref. 3 have recently been recomputed with the more exact numerical treatment of nonsimilar boundary-layer theory by Jaffe, Lind, and Smith.[5] In terms of the Levy-Lees variables used in Ref. 5, Fannelop's matching conditions with transverse curvature become

$$\frac{\Delta^*}{L} = \frac{y_\infty}{L} - (2\bar{\xi})^{1/2} \frac{f_\infty}{\rho_e u_e (r_w + y_\infty)} \tag{5}$$

and without transverse curvature

$$\frac{\Delta^*}{L} = \frac{\delta^*}{L} - (2\xi)^{1/2} \frac{f_w}{\rho_e u_e \, r_w{}^j L} \tag{6}$$

where f is the nondimensional stream function, and

$$\xi = \int_0^x \rho_e u_e \mu_e \, r_w{}^{2j} \, dx \qquad \bar{\xi} = \int_0^x \rho_e u_e \mu_e \left(\frac{r_w}{L}\right)^2 dx$$

are the transformed x variables, and L is the characteristic length.

Table 1 Boundary-layer properties for sharp cones at $x/L = 0.9$ under low-density conditions

M_∞	θ_c, deg	δ^*/δ	Δ^*/δ^*	$\tan^{-1}$ $(d\delta^*/dx)$, deg	$\tan^{-1}$ (v_e/u_e), deg	$\tan^{-1}$ $(d\Delta^*/dx)$, deg
5.64	5	0.79	1.04	1.58	0.965	1.65
5.64[a]	5	0.78	...	1.68	1.01	...
10.0	9	0.73	1.06	4.06	2.12	4.39
10.0[a]	9	0.71	...	4.12	1.97	...

[a] Data from Ref. 3.

The results of the calculations at $M_\infty = 5.64$ and 10 for zero mass transfer are given in Table 1†. For comparison, the previous results of Lewis, Marchand, and Little[3] are also given. Small differences are found on comparison of the present results with the previous results of Ref. 3. Small differences are noted also between Fannelop's mass-flow defect thickness δ_a^* ($= \Delta^*$ for zero mass transfer) and the axisymmetric displacement thickness with transverse curvature δ^*. However, we again note that significant differences exist between the complete expression proposed by Li and Gross, denoted $\tan^{-1}(v_e/u_e)$, and the hypersonic approximation proposed by them, namely, $\tan^{-1}(d\delta^*/dx)$. For these conditions, the ratios Δ^*/δ^* and $\tan^{-1}(d\Delta^*/dx)/\tan^{-1}(d\delta^*/dx)$ are both about 1.05.

In conclusion, it should be noted that the good agreement obtained by Lewis, Marchand, and Little upon comparison of their experimental and numerical results with the matching condition v_e/u_e should be regarded as somewhat fortuitous in view of Fannelop's results and the numerical results presented herein. Using the correct matching conditions given by Eq. (5) or (6) would increase the viscous-induced pressure increment above that obtained in Ref. 3 based on the $d\delta^*/dx$ matching condition. In general, the numerical results given

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† Freestream and wall conditions are given in Ref. 3.

in Ref. 3 at $M_\infty = 3.93$ and 5.64 overestimated the viscous-induced pressure when based on $d\delta^*/dx$ and were in good agreement when based on v_e/u_e conditions. The approximate tangent cone theory thus appears to have overestimated the wall pressure for the conditions at $M_\infty = 3.93$ and 5.64 studied by Lewis, Marchand, and Little.

References

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